Math 612 part 3 — Čech cohomology

Note Title 10/6/2014

Idea: use MV and combinatorics of a good cover to calculate cohom.

MV: exact rows

$$\begin{tikzcd}
 & M & U \sqcup V \arrow[l] & U \cap V \arrow[l, shift left] \arrow[l, shift right] & \\
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^2(M) \arrow[r] \arrow[u] & \Omega^2(U)\oplus\Omega^2(V) \arrow[r, "\delta"] \arrow[u] & \Omega^2(U\cap V) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Omega^1(M) \arrow[r] \arrow[u, "d"] & \Omega^1(U)\oplus\Omega^1(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(U\cap V) \arrow[r] \arrow[u, "d"] & 0 \\
0 \arrow[r] & \Omega^0(M) \arrow[r] \arrow[u, "d"] & \Omega^0(U)\oplus\Omega^0(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(U\cap V) \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

recall $\delta(\omega,\eta)=\eta-\omega$.

this is an example of a double complex : write this as $C^*(\mathcal{U},\Omega^*)$.

Note $d\delta=\delta d$, $d^2=0$.

Def A double complex $K^{*,*}$ consists of abelian gps with two gradings : $K^{*,*}=\bigoplus_{i,j} K^{i,j}$ with maps $d: K^{i,j}\to K^{i,j+1}$, $\delta: K^{i,j}\to K^{i+1,j}$ such that $d^2=\delta^2=0$, $d\delta=\delta d$.

Ex $K^{*,*}=C^*(\mathcal{U},\Omega^*)$, $K^{0,j}=\Omega^j(U)\oplus\Omega^j(V)$, $K^{1,j}=\Omega^j(U\cap V)$.

General double cx:

$$\begin{tikzcd}
{} & {} & {} & \\
K^{0,1} \arrow[r] \arrow[u] & K^{1,1} \arrow[r] \arrow[u] & K^{2,1} \arrow[r] \arrow[u] & \cdots \\
K^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & K^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & K^{2,0} \arrow[r] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

Given a double cx, can get a usual cx in several ways.

- (row, δ) ; (column, d)
- ($\oplus$ rows, $\oplus\delta$) ; ($\oplus$ columns, $\oplus d$)
- <u>or</u> define K^* by $K^n = \bigoplus_{i+j=n} K^{i,j}$

  and $D : K^n \to K^{n+1}$, $D := \delta + (-1)^i d$ on $K^{i,j}$.

$$\begin{tikzcd}
K^{i,j+1} \arrow[r, "\delta"] & K^{i+1,j+1} \\
K^{i,j} \arrow[u, "(-1)^i d"] \arrow[r, "\delta"] & K^{i+1,j} \arrow[u, "(-1)^{i+1} d"']
\end{tikzcd}$$

On $K^{i,j}$,

$$D^2 = \delta^2 + (-1)^i \delta d + (-1)^{i+1} d\delta + d^2 = 0.$$

<u>Prop</u> $0 \to X^* \xrightarrow{r} Y^* \xrightarrow{\delta} Z^* \to 0$

exact seq of chain cxs. Define the <u>mapping cone</u> of $Y^* \to Z^*$ by $K^{*,*} = (Y^* \to Z^*)$:

$$\begin{tikzcd}
 & & & \\
2 & Y^2 \arrow[u] \arrow[r, "\delta"] & Z^2 \arrow[u] \arrow[r] & 0 \\
1 & Y^1 \arrow[u] \arrow[r, "\delta"] & Z^1 \arrow[u] \arrow[r] & 0 \\
0 & Y^0 \arrow[u, "d"] \arrow[r, "\delta"] & Z^0 \arrow[u, "d"] \arrow[r] & 0 \\
 & 0 & 1 &
\end{tikzcd}$$

Then

$$\boxed{H^*(K^{*,*}, D) \cong H^*(X).}$$

<u>Pf</u> $r : X^* \to Y^* \hookrightarrow K^*$ is a chain map:

$$Dr = (\delta + \underset{i=0}{(-1)^0} d) r = dr - rd$$

$\therefore$ $r : (X^*, d) \longrightarrow (K^*, D)$ descends to $r : H^*(X, d) \to H^*(K, D)$.

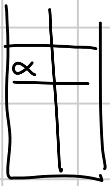
Want $\cong$.

Next note any cochain in $K^{*,*}$ is D-cohomologous to something in the 1st column: if $\alpha \in K^{*,*}$, then $\exists\, \beta$ with $\alpha - D\beta \in K^{0,*}$.

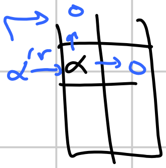
Why? $K^{0,k} = Y^k \xrightarrow{\delta} Z^k = K^{1,k} \longrightarrow 0$

$\Rightarrow \exists\, \beta$ with $\delta\beta = \alpha$ $\Rightarrow$ $\alpha - D\beta = (-1)^k d\beta \in K^{0,*}$.

<u>r^* is surjective</u>: any D-cocycle is cohom to



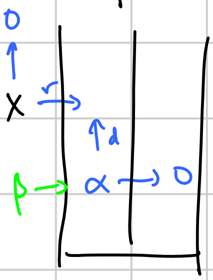
with $D\alpha = 0$



but then $\delta\alpha = 0 \Rightarrow \exists\, \alpha' \in X^*$ with $r(\alpha') = \alpha$

and $r\, d\alpha' = dr\, \alpha' = 0 \Rightarrow d\alpha' = 0 \Rightarrow \alpha'$ is closed.

<u>r^* is injective</u>: say $[x] \in H^*(X)$, $r(x) = D(\text{something}) = D(\alpha)$ for $\alpha \in Y^*$.



Then $\delta\alpha = 0$, $r(x) = d\alpha \Rightarrow \exists\, \beta \in X^*$ with $r\beta = \alpha$

$\Rightarrow r(x) = d\alpha = dr\beta = rd\beta$

$\Rightarrow x = d\beta$. □

<u>Cor</u> $H^*(C^*(\mathcal{U}, \Omega^*), D) \cong H^*_{DR}(M)$.

<u>Next</u>: generalize.

Suppose we have an exact sequence of chain cxs

$$\begin{tikzcd}
& 0 \arrow[r] & \tilde{X}^* \arrow[r, "r"] & X^{0,*} \arrow[r, "\delta"] & X^{1,*} \arrow[r, "\delta"] & X^{2,*} \arrow[r, "\delta"] & \cdots \\
\text{i.e} & & \vdots & \vdots & \vdots & & \\
& 0 \arrow[r] & \tilde{X}^1 \arrow[r, "r"] \arrow[u] & X^{0,1} \arrow[r, "\delta"] \arrow[u] & X^{1,1} \arrow[r, "\delta"] \arrow[u] & X^{2,1} \arrow[r] & \cdots \\
& 0 \arrow[r] & \tilde{X}^0 \arrow[r, "r"] \arrow[u, "d"] & X^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & X^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & X^{2,0} \arrow[r] & \cdots
\end{tikzcd}$$

↳ entire thing: augmented double complex.

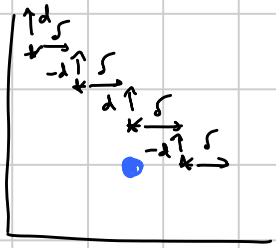
Let this be the double complex $K^{*,*}$

Prop $H^*(K^{*,*}, D) \cong H^*(\tilde{X}^*)$.

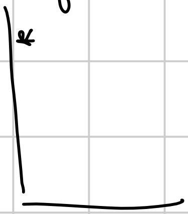
Pf As before, r gives a map $\tilde{X}^* \longrightarrow X^{0,*} \subset K^{*,*}$ descending to $r: H^*(\tilde{X}) \longrightarrow H^*(K, D)$.

- r is injective: as before.
- r is surjective:

  Any cycle in $(K^{*,*}, D)$ looks like

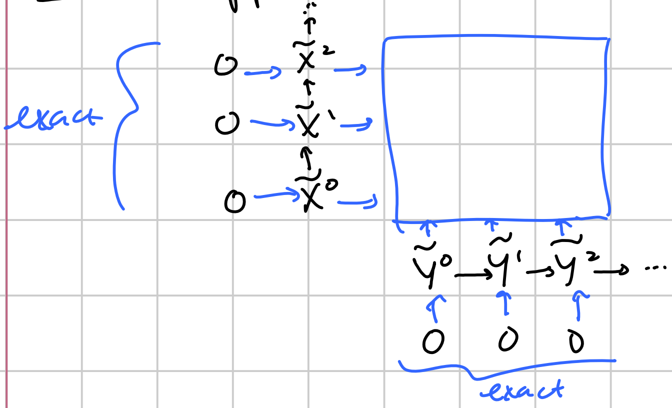


Now progressively work up the ladder. Exactness at bottom ⇒ by subtracting D(●), can get rid of the bottom piece. Etc.
So the cycle is cohomologous to



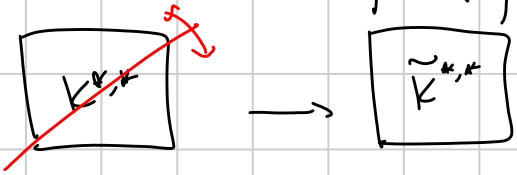
⇒ r is surjective. □

Cor Suppose we have exact rows and columns in the comm. diagram (blue)



Then
$H^*(\tilde{X}) \cong H^*(\tilde{Y})$.

Note This relies on a symmetry under reflection.



also a double cx; but $D \not\to \tilde{D}$.

Nevertheless, $H^*(K, D) \cong H^*(\tilde{K}, \tilde{D})$. (HW)

10/9

Generalized Mayer-Vietoris

Rewrite MV as: $M = U_0 \cup U_1$,

$$\begin{tikzcd} M & U_0 \amalg U_1 \arrow[l, "i"'] & U_0 \cap U_1 \arrow[l, shift left, "i_0"'] \arrow[l, shift right, "i_1"] \end{tikzcd}$$

$i_0 : U_0 \cap U_1 \to U_0$, $\partial_1 : U_0 \cap U_1 \to U_1$,

$$\rightsquigarrow \quad 0 \to \Omega^* M \xrightarrow{r = i^*} \Omega^* U_0 \oplus \Omega^* U_1 \xrightarrow{-i_0^* + i_1^*} \Omega^*(U_0 \cap U_1) \longrightarrow 0$$

Now: generalize to more open sets. (not nec. good)

$M = \cup U_\alpha$ open cover, countably many open sets U_α, index set ordered (for U_α, U_β, either $\alpha < \beta$ or $\alpha > \beta$ or $\alpha = \beta$).

For distinct $\alpha_0, \dots, \alpha_k$, define

$$U_{\alpha_0 \cdots \alpha_k} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}.$$

Then there are maps

$$\begin{tikzcd} \partial_0 : & U_{\alpha_0 \cdots \alpha_k} \arrow[r] \arrow[dr] \arrow[dddr] & U_{\alpha_1 \cdots \alpha_k} \\ \partial_1 : & & U_{\alpha_0 \alpha_2 \cdots \alpha_k} \\ & & \vdots \\ \partial_k : & & U_{\alpha_0 \cdots \alpha_{k-1}} \end{tikzcd}$$

in general, $\partial_j : U_{\alpha_0 \cdots \alpha_k} \to U_{\alpha_0 \cdots \hat{\alpha_j} \cdots \alpha_k}$.

So we get

$$\begin{tikzcd} M & \coprod_{\alpha_0} U_{\alpha_0} \arrow[l, "i"'] & \coprod_{\alpha_0 < \alpha_1} U_{\alpha_0 \alpha_1} \arrow[l, shift right, "\partial_0"'] \arrow[l, shift left, "\partial_1"] & \coprod_{\alpha_0 < \alpha_1 < \alpha_2} U_{\alpha_0 \alpha_1 \alpha_2} \arrow[l, shift right=2, "\partial_0"'] \arrow[l, "\partial_1"] \arrow[l, shift left=2, "\partial_2"] & \cdots \arrow[l, shift right=3] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3] \end{tikzcd}$$

e.g. $M \leftarrow U_0 \amalg U_1 \leftleftarrows U_{01} = U_0 \cap U_1$ (note $\partial_0 = i_1$, $\partial_1 = i_0$!)

Apply Ω^* functor:

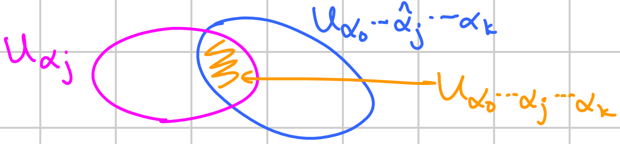
$$\begin{tikzcd} \Omega^* M \arrow[r, "r"] & \prod \Omega^*(U_{\alpha_0}) \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] & \prod_{\alpha_0 < \alpha_1} \Omega^*(U_{\alpha_0 \alpha_1}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \end{tikzcd}$$

and define $\delta_i = \partial_i^*$, $\delta = \sum (-1)^i \delta_i$.

What is this? Suppose we have $\omega \in \prod \Omega^*(U_{\alpha_0\cdots\alpha_{k-1}})$; view this as a collection of forms $\omega_{\alpha_0\cdots\alpha_{k-1}} \in \Omega^*(U_{\alpha_0\cdots\alpha_{k-1}})$. Then

$$(\delta\omega)_{\alpha_0\cdots\alpha_k} = \sum_{j=0}^{k} (-1)^j \omega_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_k}$$

(really restriction):



Prop $\delta^2 = 0$.

Pf

$$(\delta^2\omega)_{\alpha_0\cdots\alpha_{k+1}} = \sum_{j=0}^{k+1} (-1)^j (\delta\omega)_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{k+1}}$$

$$= \sum_{i<j} (-1)^{i+j} (\delta\omega)_{\alpha_0\cdots\hat{\alpha}_i\cdots\hat{\alpha}_j\cdots\alpha_{k+1}} + \sum_{i>j} (-1)^{i+j-1} (\delta\omega)_{\alpha_0\cdots\hat{\alpha}_j\cdots\hat{\alpha}_i\cdots\alpha_{k+1}}$$

$$= 0. \qquad \square$$

Rmk Compare to singular cohomology:

$$\begin{tikzcd}
\Delta_0 \arrow[r, shift left] \arrow[r, shift right] & \Delta_1 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Delta_2 & \cdots \\
C_0(X) & C_1(X) \arrow[l, shift left] \arrow[l, shift right] & C_2(X) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots \\
C^0(X) \arrow[r, shift left] \arrow[r, shift right] & C^1(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & C^2(X) & \cdots
\end{tikzcd}$$

Prop The generalized Mayer-Vietoris sequence

$$0 \to \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

is exact.

Pf We just proved this is a complex.

Exact at $\Omega^*(M)$: form on M is $0 \Leftrightarrow 0$ on each U_α.

Exactness elsewhere: we'll define a homotopy operator

$$K : \prod \Omega^*(U_{\alpha_0\cdots\alpha_k}) \longrightarrow \prod \Omega^*(U_{\alpha_0\cdots\alpha_{k-1}})$$

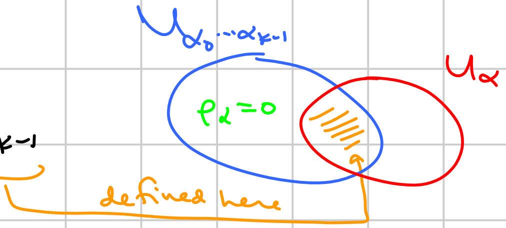
s.t. $K\delta + \delta K = \mathrm{id}$.

Choose $\{\rho_\alpha\}$ partition of unity subordinate to $\{U_\alpha\}$.
Extend the notation $\omega_{\alpha_0\cdots\alpha_k}$ from just $\alpha_0 < \cdots < \alpha_k$ to all possible indices by setting $\omega_{\cdots i \cdots j \cdots} = -\omega_{\cdots j \cdots i \cdots}$ (still defined on $U_{\alpha_0\cdots\alpha_k}$)
$\omega_{\cdots i \cdots i \cdots} = 0$.

(Formula for δ still works (exc).)

$\omega \in \prod \Omega^*(U_{\alpha_0\cdots\alpha_k})$.

Define $$(K\omega)_{\alpha_0\cdots\alpha_{k-1}} = \sum_\alpha \underbrace{\rho_\alpha}_{} \underbrace{\omega_{\alpha\alpha_0\cdots\alpha_{k-1}}}_{}$$



note $\rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{k-1}}$ is defined on $U_{\alpha_0\cdots\alpha_{k-1}}$.

$$\begin{aligned}(\delta K\omega)_{\alpha_0\cdots\alpha_k} &= \sum_j (-1)^j (K\omega)_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_k} \\ &= \sum_{\alpha,j} (-1)^j \rho_\alpha \omega_{\alpha\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_k} \\ &= \sum_\alpha \rho_\alpha \underbrace{\left(\sum_j (-1)^j \omega_{\alpha\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_k}\right)}_{-(\delta\omega)_{\alpha\alpha_0\cdots\alpha_k} + \omega_{\alpha_0\cdots\alpha_k}} \\ &= \omega_{\alpha_0\cdots\alpha_k} - (K\delta\omega)_{\alpha_0\cdots\alpha_k}. \qquad \square\end{aligned}$$

View generalized MV as an augmented double complex

$$\begin{tikzcd} 0 \arrow[r] & \tilde{X}^* \arrow[r] \arrow[d, equal] & X^{0,*} \arrow[r] \arrow[d, equal] & X^{1,*} \arrow[r] & \cdots \\ & \Omega^*(M) & \prod \Omega^*(U_{\alpha_0}) & \cdots & \end{tikzcd}$$

That is, define

$$\boxed{C^i(\mathcal{U},\Omega^j) := \prod_{\alpha_0<\cdots<\alpha_i} \Omega^j(U_{\alpha_0\cdots\alpha_i})}.$$

then we construct an augmented double cx where $K^{i,j}=C^i(\mathcal{U},\Omega^j)$:

$$\begin{tikzcd}
 & {} & {} & {} & \\
0 \arrow[r] & \Omega^2(M) \arrow[r, "r"] \arrow[u] & C^0(\mathcal{U},\Omega^2) \arrow[r, "\delta"] \arrow[u] & C^1(\mathcal{U},\Omega^2) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & \Omega^1(M) \arrow[r, "r"] \arrow[u, "d"] & C^0(\mathcal{U},\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & C^1(\mathcal{U},\Omega^1) \arrow[r] \arrow[u, "d"'] & {} \\
0 \arrow[r] & \Omega^0(M) \arrow[r, "r"] \arrow[u, "d"] & C^0(\mathcal{U},\Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & C^1(\mathcal{U},\Omega^0) \arrow[r] \arrow[u, "d"'] & K^{*,*} \\
 & 0 \arrow[u] & & & \\
 & 0 \arrow[r] & C^0(\mathcal{U},\mathbb{R}) \arrow[r, "\delta"] \arrow[uu, hook, "i"] & C^0(\mathcal{U},\mathbb{R}) \arrow[r, "\delta"] \arrow[uu, hook, "i"] & {} \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

For short:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*(M) \arrow[r] & C^*(\mathcal{U},\Omega^*) \\
 & & C^*(\mathcal{U},\mathbb{R}) \arrow[u] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

Generalized MV: rows are exact

$$\Rightarrow H^*_{DR}(M) = H^*(\Omega^*(M), d) \cong H^*(K^{*,*}, D)$$

Now define

$$C^k(\mathcal{U},\mathbb{R}) := \text{locally constant functions on each } U_{\alpha_0\cdots\alpha_k}$$
$$\subset \prod \Omega^0(U_{\alpha_0\cdots\alpha_k}) = C^0(\mathcal{U},\Omega^0).$$

Then $i: C^k(\mathcal{U},\mathbb{R}) \hookrightarrow C^k(\mathcal{U},\Omega^0)$ and $\delta: C^k(\mathcal{U},\Omega^0) \to C^{k+1}(\mathcal{U},\Omega^0)$ restricts to $\delta: C^k(\mathcal{U},\mathbb{R}) \to C^{k+1}(\mathcal{U},\mathbb{R})$.

Since $d\circ i = 0$, the (augmented) columns are now complexes.

The point: if $\mathcal{U}$ is a good cover then the columns are exact.

Poincaré lemma:

$$\begin{tikzcd}
\vdots & & & \vdots & \\
\Omega^1(\mathbb{R}^n) \arrow[u] & & & \prod \Omega^1(U_{\alpha_0\cdots\alpha_k}) \arrow[u] & \\
\Omega^0(\mathbb{R}^n) \arrow[u] & \text{exact} & \Rightarrow & \prod \Omega^0(U_{\alpha_0\cdots\alpha_k}) \arrow[u] & \text{exact.} \\
\mathbb{R} \arrow[u] & & & C^k(\mathcal{U},\mathbb{R}) \arrow[u] & \\
0 \arrow[u] & & & 0 \arrow[u] &
\end{tikzcd}$$

(Also for a good cover, locally const = const.)

Exact columns $\Rightarrow$ $H^*(K^{*,*}, D) \cong H^*(\text{bottom row})$ (use HW).

Bottom row:

$$0 \longrightarrow C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathbb{R}) \longrightarrow \cdots$$

Write $\check{H}^*(\mathcal{U}, \mathbb{R}) := H^*(C^*(\mathcal{U}, \mathbb{R}), \delta)$.

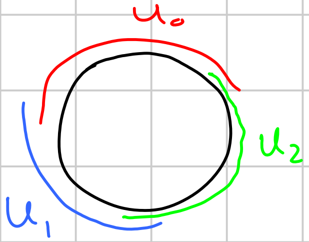
Thm If $\mathcal{U}$ is a good cover of M, then

$$H^*_{DR}(M) \cong \check{H}^*(\mathcal{U}, \mathbb{R}).$$

Cor 1. $\check{H}^*(\mathcal{U}, \mathbb{R})$ is indep of good cover.

2. If M has a finite good cover, then $H^*_{DR}(M)$ is finite-diml.

Ex 1 $M = S^1$.



$$0 \longrightarrow \underset{\substack{\shortparallel \\ \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} \\ (f_0, f_1, f_2)}}{C^0(\mathcal{U}, \mathbb{R})} \xrightarrow{\delta} \underset{\substack{\shortparallel \\ \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} \\ (g_{01}, g_{02}, g_{12})}}{C^1(\mathcal{U}, \mathbb{R})} \longrightarrow 0$$

f_i = loc const on U_i

g_{ij} = loc const on U_{ij}.

$$\delta(f_0, f_1, f_2) = (f_1 - f_0, f_2 - f_0, f_2 - f_1).$$

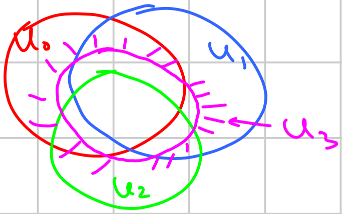
10/16 $\uparrow$ $\ker \delta = \mathbb{R}$, $\operatorname{im} \delta = \mathbb{R}^2$ $\Rightarrow$ $H^0_{DR}(S^1) = H^1_{DR}(S^1) = \mathbb{R}$.

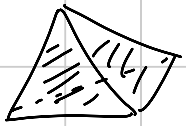
In general, to any good cover $\mathcal{U}$, can associate a simplicial complex $N(\mathcal{U})$, the nerve of $\mathcal{U}$: vertices are U_α, simplices are $(U_{\alpha_0}, \ldots, U_{\alpha_k}) \Leftrightarrow U_{\alpha_0 \cdots \alpha_k} \neq \emptyset$.

For our S^1 example, Nerve = 

and Čech complex = simplicial cochain complex for the nerve. Hmm...

Ex 2 $M=S^2$.



Nerve: 

$U_{ij} \cong \mathbb{R}^2$
$U_{ijk} \cong \mathbb{R}^2$.

$$\begin{tikzcd} 0 \arrow[r] & C^0(\mathcal{U},\mathbb{R}) \arrow[r, "\delta"] \arrow[d, phantom, sloped, "\cong"] & C^1(\mathcal{U},\mathbb{R}) \arrow[r, "\delta"] \arrow[d, phantom, sloped, "\cong"] & C^2(\mathcal{U},\mathbb{R}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\ & \mathbb{R}^4 \arrow[r, phantom, "\text{dim ker}=1"] & \mathbb{R}^6 \arrow[r, phantom, "\text{dim ker}=3"] & \mathbb{R}^4 & \end{tikzcd}$$

(f_0, f_1, f_2, f_3) $\qquad (g_{01}, \dots)$ $\qquad (h_{012}, h_{013}, h_{023}, h_{123})$

$(\delta f)_{ij} = f_j - f_i$ $\qquad (\delta g) = (g_{12}-g_{02}+g_{01},\ g_{13}-g_{03}+g_{01},\ g_{23}-g_{03}+g_{02},\ g_{23}-g_{13}+g_{12})$

$\Rightarrow \check{H}^0=\mathbb{R},\ \check{H}^1=0,\ \check{H}^2=\mathbb{R}.$

This complex is again just the simplicial cochain complex!

Generalize:

Prop $\mathcal{U}$ = good cover of M, $N(\mathcal{U})$ = nerve. Then

$$(C^*(\mathcal{U},\mathbb{R}), \delta) \underset{\uparrow \text{chain isomorphic!}}{\cong} (C^*_{simp}(N(\mathcal{U}),\mathbb{R}), \delta).$$

So $\quad \check{H}^*(\mathcal{U},\mathbb{R}) \cong H^*_{simp}(N(\mathcal{U}),\mathbb{R})$.

Pf $C^k(\mathcal{U},\mathbb{R}) \longrightarrow C^k_{simp}(N(\mathcal{U}),\mathbb{R})$:

$f \in C^k(\mathcal{U},\mathbb{R}) \Rightarrow$ have $f_{\alpha_0 \cdots \alpha_k} \in \mathbb{R}$ for $U_{\alpha_0 \cdots \alpha_k} \neq \emptyset$,

i.e. for simplex $(U_{\alpha_0}, \dots, U_{\alpha_k}) \in N(\mathcal{U})$ get $f_{\alpha_0 \cdots \alpha_k} \in \mathbb{R}$:

$f: C_k(N(\mathcal{U})) \longrightarrow \mathbb{R}$.

Furthermore, $(\delta f)_{\alpha_0 \cdots \alpha_{k+1}} = \sum (-1)^i f_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{k+1}}$

as in simplicial cohomology

$$\underset{\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_{k+1}})}{\delta f(\sigma)} = \sum (-1)^i f(\sigma|[v_0 \cdots \hat{v}_i \cdots v_{k+1}]) = \sum (-1)^i f_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{k+1}}. \quad \square$$

Now: Can go backwards as well. Say K = triangulation for M. For each simplex σ in K, recall open star

$$\mathrm{St}\,\sigma = \bigcup \mathrm{int}(\text{all simplices in } K \text{ containing } \sigma) \cup \sigma.$$

Then if $v_0, \ldots, v_k$ are vertices,

$$\bigcap \mathrm{St}\, v_i = \begin{cases} \emptyset & \text{unless } (v_0, \ldots, v_k) \text{ is a simplex in } K \\ \mathrm{St}\,\sigma & \text{if } (v_0, \ldots, v_k) = \sigma \end{cases}$$

↘ Can check: homeo (in fact diffeo) to $\mathbb{R}^n$.

So $\{\mathrm{St}\, v_i\}$ is a good cover $\mathcal{U}$ of M, and $K = N(\mathcal{U})$.

⇒ **Prop** Say M has a triangulation, simplicial cx K leading to good cover $\mathcal{U}$. Then

$$\check{H}^*(\mathcal{U}, \mathbb{R}) \cong H^*_{\mathrm{simp}}(K; \mathbb{R}).$$

⇒ **deRham Thm** M smooth (triangulable) mfd. Then

$$\boxed{H^*_{DR}(M) \cong H^*_{\mathrm{simp}}(M; \mathbb{R})} \quad (\text{since both are } \cong \check{H}^*(\mathcal{U}, \mathbb{R})).$$

If the triangulation is smooth, then

$$\underset{\omega \in \Omega^k(M)}{[\omega]} \longmapsto \Big(\underset{k\text{-simplex}}{\sigma} \longmapsto \int_\sigma \omega \Big)$$

is the isomorphism. (recall well-defined by Stokes).

} fun, tricky exercise.

Next: isomorphism preserves multiplicative structure too.

$$\begin{tikzcd} \Omega^*(M) \arrow[r, "r"] & C^*(\mathcal{U},\Omega^*) \\ & C^*(\mathcal{U},\mathbb{R}) \arrow[u, "i"'] \end{tikzcd}$$

$C^k(\mathcal{U},\Omega^\ell) = \prod_{\alpha_0<\cdots<\alpha_k} \Omega^\ell(U_{\alpha_0\cdots\alpha_k})$.

Define multiplication on $C^*(\mathcal{U},\Omega^*)$ by:

$\omega\in C^{k_1}(\mathcal{U},\Omega^{\ell_1})$, $\eta\in C^{k_2}(\mathcal{U},\Omega^{\ell_2})$ $\rightsquigarrow$ $\omega\cup\eta\in C^{k_1+k_2}(\mathcal{U},\Omega^{\ell_1+\ell_2})$:

$$(\omega\cup\eta)_{\alpha_0\cdots\alpha_{k_1+k_2}} = (-1)^{\ell_1 k_2}\,\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge\eta_{\alpha_{k_1}\cdots\alpha_{k_1+k_2}}, \qquad \alpha_0<\cdots<\alpha_{k_1+k_2}.$$

Prop. 1. $D(\omega\cup\eta) = (D\omega)\cup\eta + (-1)^{|\omega|}\omega\cup(D\eta)$

2. $r(\omega\wedge\eta) = r(\omega)\cup r(\eta)$.

Consequence: $\cup$ descends to multiplication on

$H^*(C^*(\mathcal{U},\Omega^*),D)$

and the induced map

$$\begin{tikzcd} (H^*_{DR}(M),\wedge) \arrow[r, "r"] & (H^*(C^*(\mathcal{U},\Omega^*),D),\cup) \text{ is a \underline{ring} isomorphism.} & \\ & \check{H}^*(\mathcal{U},\mathbb{R}) \arrow[u, "i"'] & \leftarrow\text{— what about this?} \end{tikzcd}$$

Note $\cup$ on $C^*(\mathcal{U},\Omega^*)$ also restricts to $\cup$ on $C^*(\mathcal{U},\mathbb{R})$:

$\omega\in C^{k_1}(\mathcal{U},\mathbb{R})\subset C^{k_1}(\mathcal{U},\Omega^0)$, $\eta\in C^{k_2}(\mathcal{U},\mathbb{R})\subset C^{k_2}(\mathcal{U},\Omega^0)$:

$$(\omega\cup\eta)_{\alpha_0\cdots\alpha_{k_1+k_2}} = \omega_{\alpha_0\cdots\alpha_{k_1}}\eta_{\alpha_{k_1}\cdots\alpha_{k_1+k_2}}:$$

Cup product in simplicial cohomology!

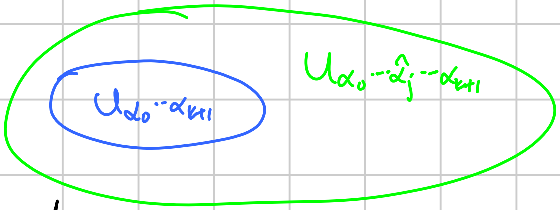
So $(\check{H}^*(\mathcal{U},\mathbb{R}),\cup) \cong (H^*_{simp}(M;\mathbb{R}),\cup)$.

Cor The isom in deRham Thm: $H^*_{DR}(M)\cong H^*_{simp}(M,\mathbb{R})$ is a ring isom.

Presheaves

The key to defining Čech cohomology: the map
$$\delta : \prod \Omega^*(U_{\alpha_0\cdots\alpha_k}) \longrightarrow \prod \Omega^*(U_{\alpha_0\cdots\alpha_{k+1}}).$$

This comes from restriction maps $\Omega^*(U_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{k+1}}) \to \Omega^*(U_{\alpha_0\cdots\alpha_{k+1}})$:



So at heart, comes from
$U \subset V \rightsquigarrow \Omega^*(V) \to \Omega^*(U)$.

More abstractly:

Def X = top'l space. A presheaf on X is a contravariant functor
$$\mathcal{F} : \mathrm{Open}(X) \longrightarrow \mathrm{Ab}$$

where morphisms in Open(X) are inclusions; i.e.,
$U \rightsquigarrow \mathcal{F}(U)$ abelian gp and if $i_{U,V} : U \hookrightarrow V$ then get
$$\mathcal{F}(i_{U,V}) = r_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U) \quad \text{s.t.}$$
$$(\sigma \longmapsto \sigma|_U)$$

① $\mathcal{F}(i_{U,U}) = r_{U,U} = id$

② $\mathcal{F}(i_{U,V})\,\mathcal{F}(i_{V,W}) = \mathcal{F}(i_{U,W})$ i.e. $r_{V,U}\, r_{W,V} = r_{W,U}$.

Ex
- Ω^* : $U \rightsquigarrow \Omega^*(U)$ (or $\Omega^k(U)$, fixed k) e.g. $C^\infty(U,\mathbb{R})$
- closed or exact forms
- X = cx mfd $\Rightarrow$ $U \rightsquigarrow \mathcal{O}(U)$ holomorphic functions

- G = abelian gp, $U \rightsquigarrow \overset{\mathcal{F}(U) =}{\{\text{locally constant functions } U \to G\}}$

This is the <u>constant presheaf</u> with group G: write presheaf as $\underline{G}$.

importantly: $\mathbb{R}, \mathbb{Z}$.

Why not $\mathcal{F}(U) = G$?

1. Constant sheaf $M \times G$, G has discrete topology: sheaf of sections is this.
2. When defining Čech, nice to have $\mathcal{F}(\phi) = 0$.
3. Actually doesn't matter for good covers.

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Extra axiom to be a sheaf:

<u>Gluing</u>: $\{U_i\}$ open cover of U, $\sigma_i \in \mathcal{F}(U_i)\ \forall i$ st $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$
$\Rightarrow \exists!\ \sigma \in \mathcal{F}(U)$ with $\sigma|_{U_i} = \sigma_i\ \forall i$.

Not a sheaf: $X = \{p_0, p_1\}$, $\mathcal{F}(\{p_0\}) = \mathbb{R}$, $\mathcal{F}(\{p_1\}) = \mathbb{R}$, $\mathcal{F}(\{p_0, p_1\}) = \mathbb{R}^3$.

<u>Def</u> A <u>homomorphism</u> of presheaves is a natural transformation of functors: $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ st.

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "r_{V,U}"'] & \mathcal{G}(V) \arrow[d, "r_{V,U}"] \\
\mathcal{F}(U) \arrow[r, "\varphi_U"] & \mathcal{G}(U)
\end{tikzcd}$$

Commutes.

Now: use a presheaf to define Čech cohomology.

$\mathcal{U}$ = open cover of top'l space X, presheaf $\mathcal{F}$. For $k \geq 0$, define

$$C^k(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_k} \mathcal{F}(U_{\alpha_0 \cdots \alpha_k})$$

<u>k-cochains on $\mathcal{U}$ with values in $\mathcal{F}$</u>.

$\omega \in C^k(\mathcal{U},\mathcal{F}) \Rightarrow \omega_{\alpha_0 \cdots \alpha_k}$ Extend by skew-symmetry as before.

Also as before, we have inclusion maps

$$\begin{tikzcd}
\coprod U_{\alpha_0 \cdots \alpha_{k+1}} \arrow[r, shift left=3, "\partial_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\partial_{k+1}"'] & \coprod U_{\alpha_0 \cdots \alpha_k}
\end{tikzcd}$$

inducing maps

$$\begin{tikzcd}
\prod \mathcal{F}(U_{\alpha_0 \cdots \alpha_k}) \arrow[r, shift left=3, "\mathcal{F}(\partial_0)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\mathcal{F}(\partial_{k+1})"'] \arrow[d, equal] & \prod \mathcal{F}(U_{\alpha_0 \cdots \alpha_{k+1}}) \arrow[d, equal] \\
C^k(\mathcal{U},\mathcal{F}) & C^{k+1}(\mathcal{U},\mathcal{F})
\end{tikzcd}$$

and define $\delta : C^k(\mathcal{U},\mathcal{F}) \to C^{k+1}(\mathcal{U},\mathcal{F})$ by $\delta = \sum_{i=0}^{k+1} (-1)^i \mathcal{F}(\partial_i)$.
Then $\delta^2 = 0$ as before.

<u>Def</u> $\check{H}^*(\mathcal{U},\mathcal{F}) = H\left(C^*(\mathcal{U},\mathcal{F}), \delta\right) =$ <u>Čech cohomology of $\mathcal{U}$ with values in $\mathcal{F}$</u>.

<u>Ex</u> $\mathcal{F} = \underline{\mathbb{R}}$: $\check{H}^*(\mathcal{U},\underline{\mathbb{R}})$ is what we previously called $\check{H}^*(\mathcal{U},\mathbb{R})$.

<u>Note</u> A homomorphism φ of presheaves $\mathcal{F} \to \mathcal{G}$ induces a chain map $(C^*(\mathcal{U},\mathcal{F}),\delta) \to (C^*(\mathcal{U},\mathcal{G}),\delta)$:

$$\begin{tikzcd}
\mathcal{F}(U_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{k+1}}) \arrow[r, "\varphi"] \arrow[d, "\mathcal{F}(\partial_i)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{k+1}}) \arrow[d, "\mathcal{G}(\partial_i)"] \\
\mathcal{F}(U_{\alpha_0 \cdots \alpha_{k+1}}) \arrow[r, "\varphi"] & \mathcal{G}(U_{\alpha_0 \cdots \alpha_{k+1}})
\end{tikzcd}$$

Thus we get a map $\check{H}^*(\mathcal{U},\mathcal{F}) \to \check{H}^*(\mathcal{U},\mathcal{G})$.

Special case: $\mathcal{F} \cong \mathcal{G}$: then C^*, $\check{H}^*$ are isomorphic.

Note $\check{H}^*(\mathcal{U},\mathcal{F})$ depends on $\mathcal{U}$ in general. But for $\mathcal{F}=\underline{\mathbb{R}}$, $\check{H}^*(\mathcal{U},\underline{\mathbb{R}})$ is indep of open cover $\mathcal{U}$ as long as $\mathcal{U}$ is good.

In general, what's the relation between $\check{H}^*(\mathcal{U},\mathcal{F})$ and $\check{H}^*(\mathcal{V},\mathcal{F})$ if $\mathcal{U},\mathcal{V}$ are open covers? Usually nothing. BUT:

Def $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$, $\mathcal{V}=\{V_\beta\}_{\beta\in J}$ open covers of M.
Then $\mathcal{V}$ is a refinement of $\mathcal{U}$, $\mathcal{U}<\mathcal{V}$, if $\exists$ map $\varphi: J\to I$ s.t.

$$V_\beta \subset U_{\varphi(\beta)} \quad \forall \beta\in J.$$

Prop Any open cover has a good refinement.

Pf Shrink the geodesically convex nbds to lie inside the U_α's, open cover $=\{U_\alpha\}$. □

If $\mathcal{U}<\mathcal{V}$, then φ induces a map

$$\varphi^\#: \underset{\prod \mathcal{F}(U_{\alpha_0\cdots\alpha_k})}{\overset{}{\check{C}^k(\mathcal{U},\mathcal{F})}} \longrightarrow \underset{\prod \mathcal{F}(V_{\beta_0\cdots\beta_k})}{\check{C}^k(\mathcal{V},\mathcal{F})}$$

$$\omega \longmapsto (\varphi^\#\omega)_{\beta_0\cdots\beta_k} = r_{U_{\varphi(\beta_0)\cdots\varphi(\beta_k)},V_{\beta_0\cdots\beta_k}}\,\omega_{\varphi(\beta_0)\cdots\varphi(\beta_k)}$$

Prop 1. $\delta\varphi^\# = \varphi^\# \delta$

2. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ in two ways $\varphi, \psi : J \to I$, then $\varphi^\#, \psi^\#$ are chain homotopic: $\exists K : C^*(\mathcal{U}, \mathcal{F}) \to C^{*-1}(\mathcal{V}, \mathcal{F})$ st

$$\varphi^\# - \psi^\# = K\delta + \delta K.$$

Pf. 1.

$$\begin{aligned}(\delta\varphi^\#\omega)_{\beta_0 \cdots \beta_{k+1}} &= \sum (-1)^i (\varphi^\#\omega)_{\beta_0 \cdots \hat{\beta_i} \cdots \beta_{k+1}} \\ &= \sum (-1)^i \, \omega_{\varphi(\beta_0) \cdots \widehat{\varphi(\beta_i)} \cdots \varphi(\beta_{k+1})} \\ &= (\varphi^\# \delta\omega)_{\beta_0 \cdots \beta_{k+1}}.\end{aligned}$$

2. HW. $\square$

So: if $\mathcal{U} < \mathcal{V}$ then $\varphi^\#$ induces a map $\check{H}^*(\mathcal{U}, \mathcal{F}) \to \check{H}^*(\mathcal{V}, \mathcal{F})$ well-defined independent of φ.

To get something indep of cover, we'd like to take a "limit" as the cover gets finer: direct limit.

Def A direct system of groups is:

- a directed set $(\mathcal{I}, \leq)$: $\leq$ is a reflexive, transitive relation, and every pair of elts $i, j \in \mathcal{I}$ has an upper bound: $\exists k \in \mathcal{I}$ with $k \geq i$, $k \geq j$.
- an abelian group G_i for each $i \in \mathcal{I}$
- homoms $\varphi_{ij} : G_i \to G_j$ for all $i \leq j$ st. $\varphi_{ii} = \mathrm{id}$, $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.

Def The *direct limit* $G = \varinjlim G_i$ is the group G satisfying the universal property:

- $\exists$ homom $\varphi_i : G_i \to G$ $\forall i$ with
$$\begin{tikzcd} G_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"'] & & G_j \arrow[dl, "\varphi_j"] \\ & G & \end{tikzcd}$$
commuting
- if $\exists$ $\psi_i : G_i \to H$ for some group H with same property, then $\exists!$ homom. $\psi : G \to H$ st.
$$\begin{tikzcd} & G_i \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_i"] & \\ G \arrow[rr, "\psi"] & & H \end{tikzcd}$$
Commutes.

Unique up to $\cong$ by construction.

Concrete construction: take $G = \left(\coprod_{i \in \mathcal{I}} G_i\right)/\sim$

$(g_i \in G_i) \sim (g_j \in G_j) \iff \exists\, k \geq i,j$ with $\varphi_{ik}(g_i) = \varphi_{jk}(g_j)$.

$[g_i] + [g_j] = [\varphi_{ik}(g_i) + \varphi_{jk}(g_j)]$ for any $k \geq i,j$.

Now: if $\mathcal{I} = \{\text{open covers}\}$, relation $<$ given by refinement, then $G_{\mathcal{U}} = \check{H}^*(\mathcal{U}, \mathcal{F})$ forms a direct system of groups.

Def The *Čech cohomology of M with values in $\mathcal{F}$* is

$$\check{H}^*(M, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^*(\mathcal{U}, \mathcal{F}).$$

Note: any homom $\mathcal{F} \to \mathcal{G}$ induces $\check{H}^*(M, \mathcal{F}) \to \check{H}^*(M, \mathcal{G})$ so if $\mathcal{F} \cong \mathcal{G}$ then $\check{H}^*(M, \mathcal{F}) \cong \check{H}^*(M, \mathcal{G})$.

In our case, do we need to take this limit? E.g. we know $\check{H}^*(\mathcal{U}, \underline{\mathbb{R}})$ indep of $\mathcal{U}$ *if* it's a good cover.

Def $\mathcal{J} \subset \mathcal{I}$ is *cofinal* if $\forall\, i \in \mathcal{I}\ \exists\, j \in \mathcal{J}$ with $i \leq j$.

Ex {good open covers} is cofinal in {open covers}.

Straightforward to check from def:

1. $$\varinjlim_{i \in \mathcal{I}} G_i = \varinjlim_{j \in \mathcal{J}} G_j$$

2. if $G_j \cong G\ \forall j \in \mathcal{J}$ in a way compatible with directed system, i.e. $\varphi_j : G_j \xrightarrow{\cong} G$ st.

$$\begin{tikzcd}
G_j \arrow[rr, "\varphi_{jj'}"] \arrow[dr, "\varphi_j"'] & \circlearrowright & G_{j'} \arrow[dl, "\varphi_{j'}"] \\
& G &
\end{tikzcd}$$

then

$$\varinjlim_{j \in \mathcal{J}} G_j \cong G.$$

Apply this to $\mathcal{I} = \{\text{open covers}\}$, $\mathcal{J} = \{\text{good open covers}\}$:

$$\check{H}^*(\mathcal{U}, \underline{\mathbb{R}}) \cong H^*_{DR}(M) \quad \text{if } \mathcal{U} \in \mathcal{J}.$$

Check that condition in #2 holds (HW).

Prop $\check{H}^*(M, \underline{\mathbb{R}}) \cong H^*_{DR}(M) \cong H^*_{simp}(M; \mathbb{R})$.

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What about $H^*_{simp}(M; G)$? $\mathcal{U}$ good cover $\rightsquigarrow N(\mathcal{U})$ simp. cx. We saw

$$\check{H}^*(\mathcal{U}, \underline{\mathbb{R}}) \cong H^*_{simp}(N(\mathcal{U}); \mathbb{R}).$$

Similarly: $\check{H}^*(\mathcal{U}, \underline{G}) \cong H^*_{simp}(N(\mathcal{U}); G)$

Now say K = triangulation of M. The barycentric subdivision $K_1 = sd\, K$, $K_2 = sd\, K_1, \ldots \rightsquigarrow$ open covers $\mathcal{U}_K, \mathcal{U}_{K_1}, \mathcal{U}_{K_2}, \ldots$ with $N(\mathcal{U}_{K_i}) = K_i$ and these are cofinal, and $H^*_{simp}(N(\mathcal{U}_{K_i}), G) \cong H^*_{simp}(M; G)$

$\rightsquigarrow$

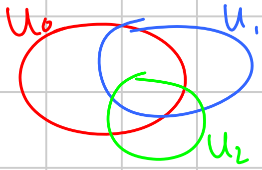
$$\boxed{\check{H}^*(M, \underline{G}) \cong H^*_{simp}(M; G)}.$$

Locally Constant Presheaves

To define $C^*(\mathcal{U},\mathcal{F})$, we don't actually need a full presheaf $\mathcal{F}$.

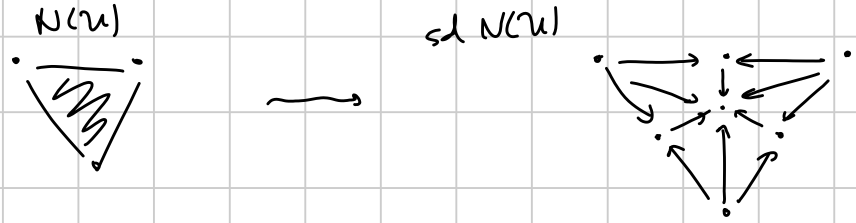
Def $\mathcal{U}$ = good cover. A presheaf on $\mathcal{U}$ is a contravariant functor
(nonempty finite intersections in $\mathcal{U}$) $\longrightarrow$ Ab
ie $\mathcal{F}(U_{\alpha_0\cdots\alpha_k})$ for each $U_{\alpha_0\cdots\alpha_k}\neq\emptyset$, and restriction maps.

ex.



$$\begin{tikzcd}
\mathcal{F}(U_0) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{F}(U_{01}) \arrow[d] & \mathcal{F}(U_1) \arrow[l] \arrow[dl] \arrow[d] \\
\mathcal{F}(U_{02}) \arrow[r] & \mathcal{F}(U_{012}) & \mathcal{F}(U_{12}) \arrow[l] \\
& \mathcal{F}(U_2) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

A useful way to think of this: $\mathcal{U} \rightsquigarrow N(\mathcal{U})$ simplicial cx.,
$sd(N(\mathcal{U}))$ = first barycentric subdivision: one vertex for each simplex.
Then a presheaf on $\mathcal{U}$ associates an abelian gp to each vertex in $sd\,N(\mathcal{U})$; edges in $sd\,N(\mathcal{U})$ are restriction maps, and the resulting diagram is commutative.



Recall constant presheaf $\underline{G}$: $\mathcal{F}(\underset{\neq\emptyset}{U_{\alpha_0\cdots\alpha_k}}) = G$, all maps = identity.

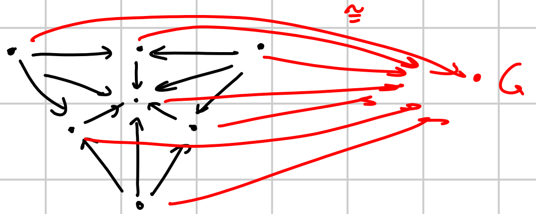
In general, might have a sheaf where $\mathcal{F}(U_{\alpha_0\cdots\alpha_k})\cong G$ but we don't know about the maps.

Def $\mathcal{U}$ = good cover. A presheaf $\mathcal{F}$ on $\mathcal{U}$ is locally constant if $\exists G$ with
$\mathcal{F}(U_{\alpha_0 \cdots \alpha_k}) \cong G \quad \forall U_{\alpha_0 \cdots \alpha_k} \neq \emptyset$ and all maps
$\emptyset \neq U \subset V \rightsquigarrow r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ are $\cong$.

Say that presheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{U}$ are isomorphic if $\exists$ natural transformation $\mathcal{F} \to \mathcal{G}$ that's an isomorphism.

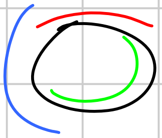
Question Are all locally constant presheaves $\cong \underline{G}$?

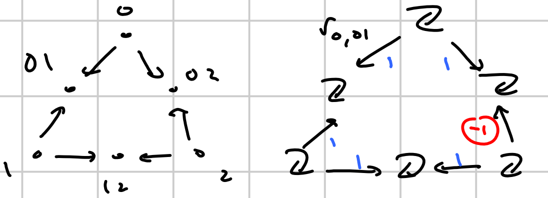
Loc. const:



Const: all triangles commute.

Answer No. S^1, $\mathcal{U}$ =



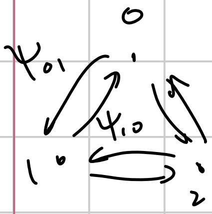


Not $\cong$ const: $\check{H}^*(\mathcal{U}, \mathcal{F}) \neq \check{H}^*(\mathcal{U}, \underline{\mathbb{Z}})$. (HW)

What's the moral reason why not? Need maps $\varphi_W : \mathcal{F}(W) \xrightarrow{\cong} \mathbb{Z}$ for each intersection W, s.t. everything commutes.

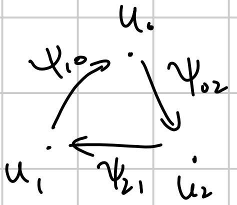
$$\begin{tikzcd}
\mathcal{F}(U_0) \arrow[r, "r_{0,01}"] \arrow[dr, "\varphi_0"', "\simeq"] & \mathcal{F}(U_{01}) \arrow[d, "\varphi_{01} = \varphi_0 r_{01}^{-1}"] & \mathcal{F}(U_1) \arrow[l, "r_{1,01}"'] \arrow[r] \arrow[dl, "\varphi_1 = \varphi_{01} r_{1,01}^{-1}"] & \mathcal{F}(U_{12}) & \mathcal{F}(U_2) \arrow[l] \arrow[r] & \mathcal{F}(U_{02}) & \mathcal{F}(U_0) \arrow[l] \\
 & \mathbb{Z} & & \cdots & & &
\end{tikzcd}$$

in general, if $U_{\alpha\beta} \neq \emptyset$ then define $\Psi_{\alpha\beta} : \mathcal{F}(U_\alpha) \to \mathcal{F}(U_\beta)$ by
$\Psi_{\alpha\beta} = r_{\beta,\alpha\beta}^{-1} \, r_{\alpha,\alpha\beta}$.



If $N(\mathcal{U})$ is connected then choosing one $\cong$ to G nails down the rest: given $\varphi_\alpha : \mathcal{F}(U_\alpha) \to \mathbb{Z}$, $\varphi_\beta = \varphi_\alpha \Psi_{\beta\alpha}$.

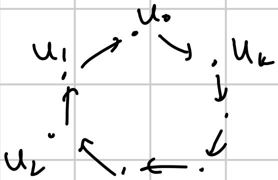
The issue:



$\varphi_0 = \varphi_0 \psi_{10}\psi_{21}\psi_{02} \Rightarrow \psi_{10}\psi_{21}\psi_{02} = id$

which isn't true in this ex!

More generally



$\psi_{10}\psi_{21}\psi_{32}\cdots\psi_{k,k-1}\psi_{0k} = id$

if $\mathcal{F} \cong$ const.

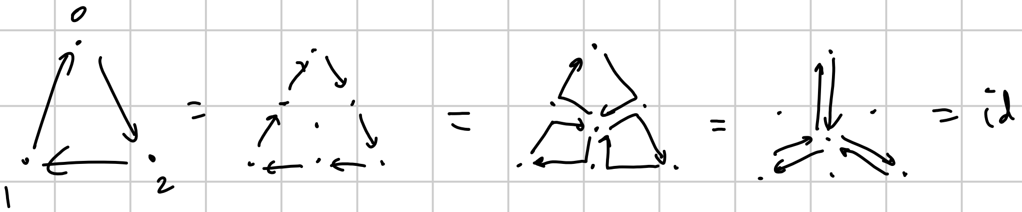
No reason this has to be true in general: instead, have map

$$\{\text{loops in } N(\mathcal{U}) \text{ at } U_0\} \longrightarrow \operatorname{Aut} G$$

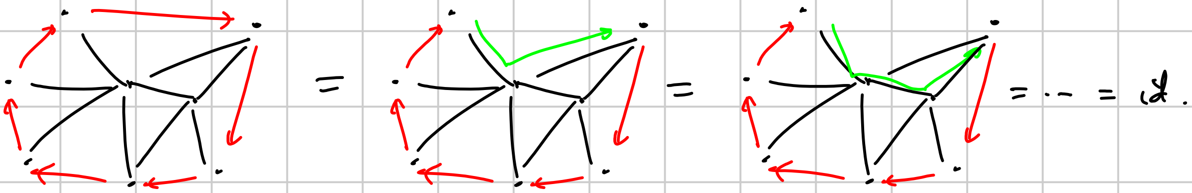
monodromy of $\mathcal{F}$

and we want this to be trivial.

Note though if (U_0, U_1, U_2) is a 2-simplex i.e. $U_{012} \neq \emptyset$ then



More generally, if the loop bounds a 2-chain then the map is id:



Prop If the monodromy is trivial — in particular, if $\pi_1(N(\mathcal{U})) = 1$ or $\operatorname{Aut} G = 1$ (eg. $G = \mathbb{Z}/2$) — then any locally constant presheaf on $\mathcal{U}$ is constant.

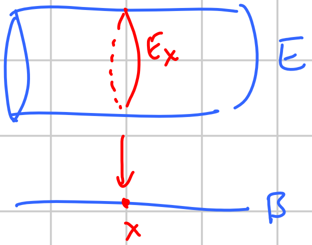
10/28 Important case of a locally const presheaf comes from fiber bundles.

Fiber Bundles

Def F, E, B smooth mfds. $\begin{matrix} E \\ \pi\downarrow \\ B\end{matrix}$ surjection is a fiber bundle with fiber F if $\exists$ open cover $\{U_\alpha\}$ of B with $\pi^{-1}(U_\alpha) \xrightarrow[\varphi_\alpha]{\cong} U_\alpha \times F$ such that

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha\times F \arrow[dl, "p"] \\ & U_\alpha & \end{tikzcd}$$

commutes.



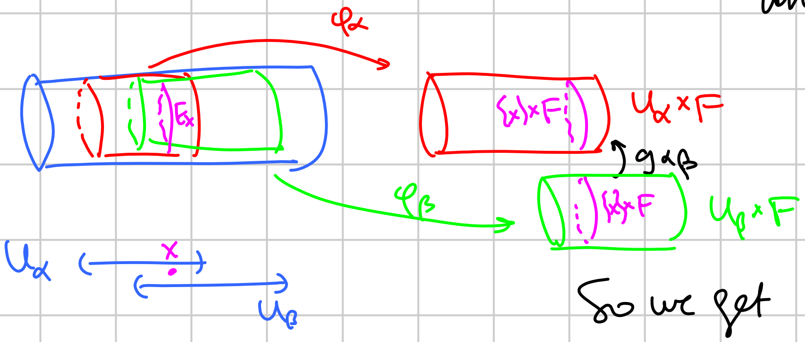
Note then we have transition fns.

$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta)\times F \to (U_\alpha\cap U_\beta)\times F$$

and in particular for $x \in U_\alpha \cap U_\beta$,

$$g_{\alpha\beta} : \{x\}\times F \to \{x\}\times F;$$

write this as $g_{\alpha\beta}(x) \in \mathrm{Diff}(F)$.



So we get $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$.
transition functions.

Note on $U_\alpha \cap U_\beta \cap U_\gamma$, $g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$: cocycle condition.

Rmk Can reconstruct the fiber bundle from B, $\{U_\alpha\}$, and transition fns $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ satisfying cocycle condition:

$$E = \coprod (U_\alpha \times F)/\sim \qquad \overset{\in U_\beta\times F}{(x,y)} \sim \overset{\in U_\alpha\times F}{(x, g_{\alpha\beta}(x)(y))}.$$

Ex 1. $E = F \times B$

2. Hopf fibration. $S^3 = \{ |z_0|^2 + |z_1|^2 = 1 \} \subset \mathbb{C}^2_{z_0, z_1}$

Define $\pi : S^3 \to \mathbb{CP}^1$ $\quad \pi(z_0, z_1) = (z_0 : z_1)$.

Then $\pi(z_0, z_1) = \pi(w_0, w_1) \Leftrightarrow w_i = \lambda z_i$ and $|\lambda| = 1$,

so $\pi^{-1}(z_0 : z_1) \cong S^1$.

$$\Rightarrow \quad \begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & \mathbb{CP}^1 = S^2 \end{tikzcd} .$$

Can generalize to
$$\begin{tikzcd} S^1 \arrow[r] & S^{2n+1} \arrow[d] \\ & \mathbb{CP}^n \end{tikzcd} .$$

3.
$$\begin{tikzcd} SO(n-1) \arrow[r] & SO(n) \arrow[d] \\ & S^{n-1} \end{tikzcd}$$
: $SO(n)$ acts transitively on S^{n-1} = unit sphere in $\mathbb{R}^n$, and the stabilizer of a pt in S^{n-1} is $SO(n-1)$.

If $\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$ is a fiber bundle it's not necessarily the case that $H^*(E) \cong H^*(B) \otimes H^*(F)$. But it is true more generally than for the trivial bundle $E = B \times F$.

Thm (Leray-Hirsch) $\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$, B has finite good cover.

If $\exists\, e_1, \ldots, e_r \in H^*(E)$ st. for each fiber E_x, the restrictions of $e_1, \ldots, e_r$ to $H^*(E_x) \cong H^*(F)$ form a basis, then
$$H^*(E) \cong H^*(B) \otimes H^*(F) \cong H^*(B) \otimes \mathbb{R}\langle e_1, \ldots, e_r \rangle.$$

Pf ∃ map

$$\begin{tikzcd} H^*(B) \otimes \mathbb{R}\langle e_1,\dots,e_r\rangle \arrow[r] & H^*(E) \\ {[\omega]} \otimes e_i \arrow[r, mapsto] & {[\pi^*\omega]} \wedge e_i . \end{tikzcd}$$

Then follow proof of Künneth Thm. □

More generally? There's a Leray spectral sequence to calculate $H^*(E)$ from $H^*(B)$, $H^*(F)$. This uses the following presheaf:

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$

Define a presheaf $\mathcal{H}^*$ on B by

$$\mathcal{H}^*(U) = H^*(\pi^{-1}(U)) , \quad U \subset B \text{ open}.$$

If $U \subset V$ then $\pi^{-1}(U) \subset \pi^{-1}(V) \Rightarrow r_{V,U} : \mathcal{H}^*(V) \to \mathcal{H}^*(U)$.

Suppose $\mathcal{U}$ is a good cover of B, and a refinement of $\{U_\alpha\}$ from def of fiber bundle (actually not necessary). Then

$U \in \mathcal{U} \Rightarrow \pi^{-1}(U) \cong U \times F \cong \mathbb{R}^n \times F$

(or $U = \cap$ open sets in $\mathcal{U}$) $\Rightarrow$ by Poincaré, $\mathcal{H}^*(U) = H^*(\pi^{-1}(U)) \cong H^*(F)$.

Also if $U \subset V$ then

$$\begin{tikzcd} \mathcal{H}^*(V) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{H}^*(U) \arrow[d, phantom, "\cong" rotate=90] \\ H^*(F) \arrow[r, "id"] & H^*(F) \end{tikzcd}$$

So $\mathcal{H}^*$ gives a locally constant presheaf over $\mathcal{U}$.

In general, not constant! But: choose $\mathcal{U}$ s.t. $N(\mathcal{U}) = K$, simplicial cx for B. Then if B is simply connected, so is $K = N(\mathcal{U})$, so any locally const presheaf on $\mathcal{U}$ is constant.